LOYOLA COLLEGE (AUTONOMOUS) CHENNAI 600 034

M. Sc DEGREE EXAMINATION – Mathematics

First Semester – November 2013

MT 1816 – Real Analysis Max: 100 Marks

Time: Forenoon/Afternoon Date:

Answer **ALL** Questions. All question carry equal marks.

1. a) (i) If f is Riemann integrable on [a, b] and $F(x) = \int_a^x f(t)dt$, for $a \le x \le b$, then prove that F is continuous on [a, b]. Further if f is continuous at x_0, where $x_0 \in [a,b]$, then prove that F is differentiable and $F'(x_0)=f(x_0)$.

OR

(ii) If f_1 □ □(α) and $f_2 \in$ □(α) on $[a, b]$, then prove that $f_1 + f_2$□ □(α). (5)

b) (i) Prove that $f \in \Re(\alpha)$ on [a, b] if and only if for every $\epsilon > 0$, there exists a partition **P** such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$.

(ii) If f is monotonic on [a, b] and if α is continuous on [a, b], then prove that $f \in$ □(α). (8+7)

OR

(iii) Assume α increases monotonically and $\alpha' \in \Re$ on [a, b]. Let f be a bounded real function on [a,b]. Then prove that $f \in$ □(α) if and only if $f\alpha' \in$ □. In that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

(iv) State and prove the fundamental theorem of calculus. (9+6)

2. (a) (i) Let α be monotonically increasing on $[a,b]$, then $f_n \varepsilon$□(α) on $[a,b]$, for $n = 1,2,3,...$ and suppose $f_n \to f$ on $[a,b]$. Then prove that $f \varepsilon$□(α) on $[a,b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$. (5)

OR

(ii) Illustrate with an example that the limit of the integral need not be equal to the integral of the limit. (5)

(P.T.O)

(b) (i) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f'_n\}$ converges uniformly on $[a,b]$, then prove that $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and $f'(x)=\lim_{n\to\infty} f'_n(x) (a\le x\le b)$.

(ii) Prove that the sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\varepsilon>0$ there exists an integer N such that $m\ge N$, $n\ge N$, $x\varepsilon E$ implies $|f_n(x)-f(x)|\le\varepsilon$. (8+7)

OR

(iii) State and prove Stone- Weierstrass theorem. (15)

3 a) (i) Let $=\{\varphi_0,\varphi_1,\varphi_2,...\}$ be orthnormal on I and assume that $f\in L^2(I)$. Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ on I as follows: $s_n(x)=\sum_{k=0}^{\infty} c_k\varphi_k(x)$, $t_n(x)=\sum_{k=0}^{\infty} b_k\varphi_k(x)$ where $c_k=(f,\varphi_k(x)$ for k = 0, 1, 2... and b_0, b_1, b_2 ... are arbitrary complex numbers. Then for each n, prove that $\|f-s_n\|\le\|f-t_n\|$.

OR

(ii) State and prove Parseval's formula. (5)

b) (i) State and prove Riesz-Fischer theorem.

(ii) Prove that for each real β and f ∈L(I), $\lim_{\alpha\to\infty}\int_I f(t)\sin(\alpha t+\beta)\,dt=0$. (9+6)

OR

(iii) If g is of bounded variation on $[0,\delta]$, then prove that $\lim_{\alpha\to\infty}\frac{2}{\pi}\int_0^{\delta} g(t)\frac{\sin\alpha t}{t}\,dt=g(0+)$.

(iv) Assume that $f\in L[0,2\pi]$ and suppose that f is periodic with period 2π. Let $\{s_n\}$ denote the sequence of partial sums of the Fourier series generated by f, $s_n(x)=\frac{a_0}{2}+\sum_{k=0}^{\infty}(a_k\cos kx+(b_k\sin kx)$, n=1,2,...Then prove that $s_n(x)=\frac{2}{\pi}\int_0^{\pi}\frac{f(x+t)+f(x-t)}{2}D_n(t)\,dt$ where D_n is called Dirichlet's Kernel. (8+7)

4. a) (i) If A, B∈L(R^n, R^m)and c is a scalar, then prove that, $\|A+B\|\le\|A\|+\|B\|$ and $\|cA\|=|c|\|A\|$.

OR

(ii) State and prove the fixed point theorem for a complete metric space. (5)

b) (i) State and prove the inverse function theorem.

OR

(ii) State and prove the implicit function theorem. (15)

5.(a)(i) Graph the circle given by $x^2 + y^2 = 1$. Using the graphical approach, determine parts of the graph that have inverses and algebraic approach, find invertible formulas and cases converting x to y.

OR

(ii) Suppose that a cup of tea starts out at 98°C and that the conference room you are in is at a temperature of 72°C and suppose that after three minutes, the tea temperature has dropped to 90°C. The conference session is to go on for some time. How long will it take for tea to cool down to 80°C? (5)

(b) (i) Derive D' Alembert's approach toward characterizing solutions of the vibrating string .

OR

(ii)Derive the heat equation. (15)

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